

Section 9.3: Separable Equations.

An ordinary differential equation (ODE) is an Equation that relates a function $y = f(x)$, to its derivatives, and to x .

Example: $y''(x) - 2y'(x) + y(x) - x^2 = 0$ is a 2^{nd} order ODE
(The order of an ODE is the highest derivative in it)

Definition: A separable equation is a 1^{st} order ODE, in which $y'(x) = \frac{dy}{dx}$ can be separated as $y'(x) = \frac{dy}{dx} = g(x) \cdot f(y)$

We could also write $\frac{dy}{f(y)} = g(x) dx$ and thus $\int \frac{dy}{f(y)} = \int g(x) dx$

Examples ① Solve the Separable equation $y^3 \cdot y' = x^3$

We can rewrite this ODE as $\frac{dy}{dx} = \frac{x^3}{y^3}$. Further, we have $y^3 dy = x^3 dx \Rightarrow$ Integrate Separately

$$\int y^3 dy = \int x^3 dx \Rightarrow \frac{1}{4}y^4 = \frac{1}{4}x^4 + C$$

$$\Rightarrow y^4 = x^4 + 4C \Rightarrow y = \pm \sqrt[4]{x^4 + 4C}$$

This means that $y = \pm \sqrt[4]{x^4 + 4C}$ satisfies $y^3 \cdot y' = x^3$ for any choice of C .

Suppose we know, in addition, that the solution $y(x)$ satisfies $y(0) = -2$. Can we find C ?

Well, $y(0) = \pm \sqrt[4]{0^4 + 4C} = -2$. The negative sign in -2 indicates that we should have $-\sqrt[4]{\cdot}$, not $\sqrt[4]{\cdot}$

So, $y(0) = -\sqrt[4]{4C} = -2$, which implies that $4C = 16$

Finally, we conclude that $y(x) = -\sqrt[4]{x^4 + 16}$.

② Solve the "initial value problem" $e^{2x+y} \cdot y' = 1$, $y(0) = 0$

$$e^{2x+y} \frac{dy}{dx} = 1 \Rightarrow e^{2x} \cdot e^y dy = dx \Rightarrow e^y dy = \frac{dx}{e^{2x}}$$

$$\Rightarrow e^y dy = e^{-2x} dx \rightarrow \text{Integrate.}$$

$$\int e^y dy = \int e^{-2x} dx \quad \text{let } u = -2x, \quad -\frac{du}{2} = dx.$$

$$e^y = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-2x} + C$$

Thus, $e^y = -\frac{1}{2} e^{-2x} + C$. $\underset{x}{\overset{y}{\uparrow}} \quad y(0) = 0$ implies that

$$e^0 = -\frac{1}{2} e^0 + C \Rightarrow 1 = -\frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

So, $e^y = \frac{3}{2} - \frac{1}{2} e^{-2x} \rightarrow \text{Take ln of both sides}$

Finally we get $y = \ln \left(\frac{3}{2} - \frac{1}{2} e^{-2x} \right)$.